Banach spaces with the 3.2.I.P.

Åsvald Lima

The aim of this note is to study the structure of real Banach spaces with the 3.2.I.P. We give some characterizations of Banach spaces with the 3.2.I.P. and we prove that a Banach space A has the 3.2.I.P. if and only if its dual A^* has the 3.2.I.P.

In this paper A is a real Banach space. Its closed unit ball is denoted by A_1 and if $r\geq o$ and $x\in A$, then $B(x,r)=\{a\in A:\|x-a\|\leq r\}$ A^* is the dual space of A. If $S\subseteq A$, then $co(S)$ is the convex hull of S, and if S is convex $\partial_e S$ denote the extreme points of S. A is said to have the <u>n,2 intersection property</u> (n.2.I.P.), where n is an integer with $n\geq 3$, if for every collection of n balls $\{B(a_i,r_i)\}_{i=1}^n$ in A such that $B(a_i,r_i)\cap B(a_j,r_j)\neq\emptyset$ all $i,j\in\{1,2,\ldots,n\}$, we have $\bigcap_{i=1}^n B(a_i,r_i)\neq\emptyset$.

The main tool in studying Banach spaces with the 3.2.I.P. will be the following notion. A is said to have <u>property</u> R_3 if for all $x,y\in A$, there exists $z,u,v\in A$ such that

$$(*)\quad\left\{\begin{array}{ll} x=z+u & \|x\|=\|z\|+\|u\| \\ y=z+v & \|y\|=\|z\|+\|v\| \\ & \|x-y\|=\|u-v\|=\|u\|+\|v\| \end{array}\right.$$

Before we state the first theorem we will give some more definitions. It is easy to see that the maximal convex subsets of the boundary of A_1 coincide with the maximal proper faces of A_1 and the maximal proper faces of A_1 are all normclosed. (See [1] or [3]). We follow Fullerton and call A a <u>CL space</u> if for every maximal proper face F of A_1 we have $A_1=co(F\cup -F)$.

We call C a <u>facial cone</u> if $C=\bigcup_{\lambda\geq o}\lambda F$ for some proper face F of A_1.

Let C be a facial cone and let $z,x,y\in C$ with $o\leq z\leq x,y$ (in the ordering generated by C). Then we have

$\|x\|=\|z\|+\|x-z\|$, $\|y\|=\|z\|+\|y-z\|$ and

$\|x-y\|\leq\|x-z\|+\|y-z\|$, so $2\|z\|\leq \|x+y\|-\|x-y\|$.

The facial cone C is said to be a <u>R_3 cone</u> if for all $x,y\in C$, there exists $z\in C$ with $o\leq z\leq x,y$ and $2\|z\|=\|x+y\|-\|x-y\|$.

THEOREM 1: Let A be a real Banach space. The following statements are equivalent:

(i) A has the 3.2.I.P.
(ii) A has property R_3.

Proof: (i)=>(ii). Let $x,y \in A$. Define
$r_o, r_1, r_2 \geq o$ by $2r_o = \|x\| + \|y\| - \|x-y\|$,
$2r_1 = \|x\| - \|y\| + \|x-y\|$ and $2r_2 = -\|x\| + \|y\| + \|x-y\|$
Then $\|x\| = r_o + r_1$, $\|y\| = r_o + r_2$ and $\|x-y\| = r_1 + r_2$, so
$B(o,r_o)$, $B(x,r_1)$ and $B(y,r_2)$ are
mutually intersecting. Let $z \in B(o,r_o) \cap B(x,r_1) \cap B(y,r_2)$
and define $u = x-z$ and $v = y-z$.
Then $x = z+u$ and $\|x\| \leq \|z\| + \|u\| \leq r_o + r_1 = \|x\|$
so $\|x\| = \|z\| + \|u\|$. Similarly we show that $y = z+v$, $\|y\| = \|z\| + \|v\|$ and
$\|x-y\| = \|u-v\| = \|u\| + \|v\|$.
(ii)=>(i) Assume given 3 mutually intersecting balls in A.
As in [8; Theorem 4.6] we may assume these balls are
$B(o,r_o)$, $B(x,r_1)$ and $B(y,r_2)$ where
$\|x\| = r_o + r_1$, $\|y\| = r_o + r_2$ and $\|x-y\| \leq r_1 + r_2$.
Let z, u and v be as in (*). Then we have
$o \leq r_1 - \|u\| = \|z\| - r_o = r_2 - \|v\|$. Define $w = \|z\|^{-1} r_o z$ ($w = o$ if $z = o$). Then
$\|o-w\| = r_o$, $\|x-w\| = \|z+u-\frac{r_o}{\|z\|}z\|$
$= \|u + \frac{(\|z\| - r_o)}{\|z\|} z\| \leq \|u\| + (\|z\| - r_o) = r_1$ and
$\|y-w\| = \|z+v-\frac{r_o}{\|z\|}z\| \leq \|v\| + (\|z\| - r_o) = r_2$, so
$w \in B(o,r_o) \cap B(x,r_1) \cap B(y,r_2)$.

THEOREM 2: Assume A has property R_3. Then A is a CL space and every facial cone in A is a R_3 cone.

Proof: That every facial cone in A is a R_3 cone follows easily from (*) and Lemma 2.7 in [1] .

Let F be a maximal proper face of A_1 and let $C = \bigcup_{\lambda \geq o} \lambda F$. Then C is a norm closed convex cone. Let $x \in A$. By Theorem 2.9 in [1], there exists $y \in C$ and $z \in C'$ (=the complementary cone of C; see [1]) such that

$$x=y+z, \quad \|x\|=\|y\|+\|z\|.$$

Let $w \in C$. By (*) there exists $s,u,v \in A$ such that

$$z=s+u \quad \|z\|=\|s\|+\|u\|$$

$$w=s+v \quad \|w\|=\|s\|+\|v\|$$

$$\|w-z\|=\|u-v\|=\|u\|+\|v\|$$

By Lemma 2.7. in [1] we get $s \in C \cap C' = (o)$, so $s=o$. Hence

$$\|w-z\|=\|w\|+\|z\| \quad \text{all} \quad w \in C.$$

If $z \neq o$, define $F' = co(F \cup \frac{-z}{\|z\|})$. Then F' is a convex subset of the boundary of A_1 and $F \subseteq F'$. Since F is maximal, we have $F=F'$, so $-z \in C$. But then $A_1 = co(F \cup -F)$ so A is a CL space, and the proof is complete.

Let $(A^n, \| \|_\infty) = \left\{ (a_1 \ldots a_n) \in A^n : \|(a_1 \ldots a_n)\| = \max \|a_i\| \right\}$. Then $(A^n, \| \|_\infty)$ is a Banach space whose dual space is isometric to

$$(A^{*n}, \| \|_1) = \left\{ (f_1 \ldots f_n) \in A^{*n} : \|(f_1 \ldots f_n)\| = \sum_{i=1}^{n} \|f_i\| \right\}$$

In $(A^{*n}, \| \|_1)$ we define the w^*-closed subspace
$H^n(A^*) = \{ (f_1 \ldots f_n) \in A^{*n} : \sum_{i=1}^{n} f_i = o \}$.

We will also consider the following subsets of $H^n(A^*)$:

$$S_n = \{ f \in H^n(A^*)_1 : f = (z_1 g, \ldots, z_n g) \text{ where } g \in A^* \text{ and } (z_1, \ldots, z_n) \in H^n(R) \}$$

and

$$S_{jk} = \{ (f_1, \ldots, f_n) \in H^n(A^*)_1 : f_i = o \text{ for } i \neq j \text{ and } i \neq k \}$$

where $j<k$ and $j,k \in \{1,2,\ldots,n\}$

THEOREM 3: The following statements are equivalent:

(i) A has the 3.2.I.P.

(ii) $\partial_e H^3(A^*)_1 \subseteq S_3$

(iii) $H^3(A^*)_1 = \mathrm{co}(S_{12} \cup S_{13} \cup S_{23})$

(iv) A^* has property R_3.

Proof: (i)<=>(ii) follows from Corollary 2.7 in [7] and Lemma 4.2 in [8]

(ii)=>(iii) If $(f_1,f_2,f_3) \in \partial_e H^3(A^*)_1$, then by (ii) $(f_1,f_2,f_3)=(z_1 g, z_2 g, z_3 g)$ where $g \in A^*$ and $z_1+z_2+z_3=0$. Then clearly at most two z_i can be different from 0, so $(f_1,f_2,f_3) \in S_{12} \cup S_{13} \cup S_{23}$. Since S_{jk} all are convex and w^* - compact $\partial_e H^3(A^*)_1$ is contained in the convex w^* - compact set $\mathrm{co}\,(S_{12} \cup S_{13} \cup S_{23})$ and (iii) follows.

(iii) =>(ii) is obvious.

(iii) =>(iv) Let $f_1, f_2 \in A^*$. Then $(f_1, -f_2, f_2-f_1) \in H^3(A^*)$ and we may assume $\|f_1\|+\|f_2\|+\|f_1-f_2\|=1$.
By (iii) there exists $\lambda_1,\lambda_2,\lambda_3 \geq 0$, $\sum_{i=1}^{3} \lambda i=1$, $g_1,g_2,g_3 \in A^*$ such that $\|g_i\| \leq \frac{1}{2}$ and $(f_1,-f_2,f_2-f_1)=\lambda_1(g_1,-g_1,0)+\lambda_2(g_2,0,-g_2)+\lambda_3(0,-g_3,+g_3)$
From this it follows that

$$f_1=\lambda_1 g_1 + \lambda_2 g_2 \qquad \|f_1\|=\|\lambda_1 g_1\|+\|\lambda_2 g_2\|$$

$$f_2=\lambda_1 g_1 + \lambda_3 g_3 \qquad \|f_2\| \doteq \|\lambda_1 g_1\|+\|\lambda_3 g_3\|$$

$$\|f_1-f_2\|=\|\lambda_2 g_2-\lambda_3 g_3\|=\|\lambda_2 g_2\|+\|\lambda_3 g_3\|$$

so A^* has property R_3.

(iv)=>(iii). Let $(f_1,f_2,f_3) \in H^3(A^*)_1$. By (*) there exists $g,h,k \in A^*$ such that

$$f_1=g+h \qquad \|f_1\|=\|g\|+\|h\|$$

$$-f_2=g+k \qquad \|f_2\|=\|g\|+\|k\|$$

$$\|f_1+f_2\|=\|h-k\|=\|h\|+\|k\|$$

So

$$(f_1,f_2,f_3) = (g, -g,o) + (h,o, -h) + (o, -k,k)$$
$$\in co(S_{12} \cup S_{13} \cup S_{23})$$

and the proof is complete.

COROLLARY 4: A has the 3.2.I.P. if and only if A^* has the 3.2.I.P.

COROLLARY 5: If A has the 3.2.I.P. then both A and A^* are CL spaces.

REMARK: Hanner [5] proved corollary 4 for finite dimensional spaces. Corollary 5 improves a result of Lindenstrauss [8; Theorem 4.8.].

We say that A has property R_4 if for all $x,y,z \in A$, there exists $u_1,\dots,u_6 \in A$ such that

$$(**)\quad \left\{\begin{array}{ll} x = u_1 + u_2 + u_3 & \|x\| = \|u_1\| + \|u_2\| + \|u_3\| \\ y = -u_1 + u_4 + u_5 & \|y\| = \|u_1\| + \|u_4\| + \|u_5\| \\ z = \quad -u_2 - u_4 + u_6 & \|z\| = \|u_2\| + \|u_4\| + \|u_6\| \\ \|x+y+z\| = \|u_3+u_5+u_6\| = \|u_3\| + \|u_5\| + \|u_6\| & \end{array}\right.$$

As in Theorem 3 we get:

THEOREM 6: The following statements are equivalent:

(i) A has the 4.2.I.P.

(ii) $\partial_e H^4(A^*)_1 \subseteq S_4$

(iii) $H^4(A^*)_1 = co\ (S_{12} \cup S_{13} \cup S_{14} \cup S_{23} \cup S_{24} \cup S_{34})$

(iv) A^* has property R_4.

THEOREM 7: A is isometric to an $L_1(\mu)$ space if and only if A has property R_4.

Proof: That every $L_1(\mu)$ space has property R_4 is straightforward. (Decompose the functions in positive and negative parts and use Riesz decomposition property.)

Assume now that A has property R_4. Let F be a maximal proper face of A_1 and let $C = \bigcup_{\lambda \geq o} \lambda F$. Then since A has property R_3 we have $A_1 = co(F \cup -F)$. Let $x, y \in C$. Since C is a R_3 cone, there exists $z \in C$ such that $o \leq z \leq x, y$ and

$$2\|z\| = \|x+y\| - \|x-y\|$$

Let $o \leq w \leq x + y$. If we use R_4 on x, y and $-w$ and use that C is a facial cone, then by (**) and Lemma 2.7 in [1] we find $u_2, u_3, u_4, u_5 \in C$ such that

$$x = u_2 + u_3$$

$$y = u_4 + u_5$$

$$w = u_2 + u_4$$

i.e. C has Riesz decomposition property. But then if $o \leq z, w \leq x, y$, $z, w \leq u \leq x, y$ for some $u \in C$, so we have $\|z\| = \|u\|$. Hence $u = z$, so $w \leq u = z$. This shows that $z = \text{infimum } (x,y)$, so A with positive cone C is a vector lattice, and $\inf(x,y) = o$ implies $\|x+y\| = \|x-y\| = \|x\| + \|y\|$. By a theorem of Kakutani [6], A is isometric to a $L_1(\mu)$ space.

COROLLARY 8: A has the 4.2.I.P. if and only if A^* is isometric to a $L_1(\mu)$ space.

REMARK: Corollary 8 is a well known result of Lindenstrauss [8].

THEOREM 9: A is isometric to a $L_1(\mu)$ space if and only if A^* has the 4.2.I.P.

Proof: Assume A is isometric to a $L_1(\mu)$ space. Then A has property R_4, and Lemma 2.2. in [7] gives that A^* has the 4.2.I.P.

Assume A^* has the 4.2.I.P. Then by Theorem 1 and Corollary 4, A has the R_3 property. Let F be a maximal proper face of A_1 and let $C = \bigcup_{\lambda \geq o} \lambda F$. Now we proceed as in the proof of Theorem 7 except that we prove that C has the Riesz decomposition property in the following way. Let $e \in \partial_e A^*_1$ with $e(x)=1$ all $x \in F$. Then
$$C^* = \{f \in A^* : f(x) \geq o \text{ all } x \in C\}$$
$$= \{f \in A^* : f = \lambda(e+g),\ \lambda \geq o \text{ and } g \in A^*_1\}$$
and C^* has Riesz decomposition property by Theorem 6.1.(12)=>(14) in [8]. By a result of Andô [2] it follows that C has Riesz decomposition property, and the proof is complete.

REMARK: Theorem 9 is a well known result of Grothendieck [4] and Lindenstrauss [8].

REFERENCES.

1. E. Alfsen and E. Effros: Structure in Real Banach spaces I&II Annals of Math. 96 (1972), 98-173.

2. T. Andô: On fundamental properties of a Banach space with a cone. Pacific J. Math.(2)12(1962),1163-1169.

3. R.E. Fullerton: Geometrical characterizations of certain function spaces. Proc. Inter. Symposium on Linear Spaces, Jerusalem (1961), 227-236.

4. A. Grothendieck: Une caractérisation vectorielle métrique des espaces L^1. Canadien J.Math.7(1955),552-561.

5. O. Hanner: Intersection of translates of convex bodies. Math. Scand. 4(1956),65-87.

6. S. Kakutani: Concrete representation of abstract L spaces and the mean ergodic theorem. Ann. of Math. 42 (1941),523-537.

7. Å. Lima: Intersection properties of balls and subspaces in Banach spaces. (1974).

8. Lindenstrauss: Extensions of compact operators. Memoirs Amer. Math. Soc.48(1964).